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# Poincaré Supersymmetry Representations Over Trace Class Noncommutative Graded Operator Algebras

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## ABSTRACT

We show that rigid supersymmetry theories in four dimensions can be extended to give supersymmetric trace (or generalized quantum) dynamics theories, in which the supersymmetry algebra is represented by the generalized Poisson bracket of trace supercharges, constructed from fields that form a trace class noncommutative graded operator algebra. In particular, supersymmetry theories can be turned into supersymmetric matrix models this way. We demonstrate our results by detailed component field calculations for the Wess-Zumino and the supersymmetric Yang-Mills models (the latter with axial gauge fixing), and then show that they are also implied by a simple and general superspace argument.

## 1. Introduction to Trace Dynamics

In constructing supersymmetric field theories, one usually verifies the supersymmetry by doing a *classical* Grassmann calculation, treating the bosonic fields as classical (rather than operator) variables and the fermionic fields as classical Grassmann (rather than operator Grassmann) variables. Then one quantizes by replacing the classical Poisson or Dirac brackets by commutators/anticommutators. We shall show in this paper that for rigid supersymmetry theories, a significant generalization of this standard approach is possible, in which the pre-quantum bosonic and fermionic fields are respectively trace class even and odd grade *operators*, such as, for example,  $N \times N$  matrices whose matrix elements are respectively the even and odd grade elements of a complex Grassmann algebra. In particular, our results show that rigid supersymmetry theories can be extended to give supersymmetric matrix models. The requirement that the field variables be of trace class is crucial to our results, since in the calculations given below, cyclic permutation of operator variables under the trace provides the necessary commutativity, generalizing the trivial commutativity/anticommutativity of classical field variables, for verifying both supersymmetry of the Lagrangian and the closure of the supersymmetry algebra.

Our approach is based on the trace (or generalized quantum) dynamics that we have proposed [1] and studied with various collaborators [2]; we in fact shall use a simplified form of this dynamics that becomes possible when Grassmann algebras are employed to represent the fermion/boson distinction. Let  $B_1$  and  $B_2$  be two  $N \times N$  matrices with matrix elements that are even grade elements of a complex Grassmann algebra, and  $\text{Tr}$  the ordinary matrix

trace, which obeys the cyclic property

$$\text{Tr} B_1 B_2 = \sum_{m,n} (B_1)_{mn} (B_2)_{nm} = \sum_{m,n} (B_2)_{nm} (B_1)_{mn} = \text{Tr} B_2 B_1 \quad . \quad (1a)$$

Similarly, let  $\chi_1$  and  $\chi_2$  be two  $N \times N$  matrices with matrix elements that are odd grade elements of a complex Grassmann algebra, which anticommute rather than commute, so that the cyclic property for these takes the form

$$\text{Tr} \chi_1 \chi_2 = \sum_{m,n} (\chi_1)_{mn} (\chi_2)_{nm} = - \sum_{m,n} (\chi_2)_{nm} (\chi_1)_{mn} = -\text{Tr} \chi_2 \chi_1 \quad . \quad (1b)$$

The cyclic/anticyclic properties of Eqs. (1a, 1b) are just those assumed for the trace operation **Tr** of trace dynamics, although in Refs. [1, 2] the fermionic operators were realized as matrices with complex matrix elements, all of which anticommute with a grading operator  $(-1)^F$  which formed part of the definition of **Tr**. Since the use of Grassmann odd fermions eliminates the need for the inclusion of the  $(-1)^F$  factor \*, we shall continue here to use the notation **Tr**, with the understanding that fermionic matrices or operators obey Eq. (1b) while bosonic matrices or operators obey Eq. (1a). From Eqs. (1a, b), one immediately derives the trilinear cyclic identities

$$\begin{aligned} \text{Tr} B_1 [B_2, B_3] &= \text{Tr} B_2 [B_3, B_1] = \text{Tr} B_3 [B_1, B_2] \\ \text{Tr} B_1 \{B_2, B_3\} &= \text{Tr} B_2 \{B_3, B_1\} = \text{Tr} B_3 \{B_1, B_2\} \\ \text{Tr} B \{\chi_1, \chi_2\} &= \text{Tr} \chi_1 [\chi_2, B] = \text{Tr} \chi_2 [\chi_1, B] \\ \text{Tr} \chi_1 \{B, \chi_2\} &= \text{Tr} \{\chi_1, B\} \chi_2 = \text{Tr} [\chi_1, \chi_2] B \quad , \end{aligned} \quad (1c)$$

which are repeatedly used below.

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\* If the  $(-1)^F$  construction is combined with Grassmann odd fermions one gets the “supertrace”  $\text{str}$ , that obeys the cyclic property  $\text{str} N_1 N_2 = \text{str} N_2 N_1$  for both bosonic and fermionic  $N_{1,2}$ . We will not use the supertrace in this article.

The basic observation of trace dynamics is that given the trace of a polynomial  $P$  constructed from noncommuting matrix or operator variables, one can define a derivative of the  $c$ -number  $\text{Tr}P$  with respect to an operator variable  $\mathcal{O}$  by varying and then cyclically permuting so that in each term the factor  $\delta\mathcal{O}$  stands on the right, giving the fundamental definition

$$\delta\text{Tr}P = \text{Tr}\frac{\delta\text{Tr}P}{\delta\mathcal{O}}\delta\mathcal{O} \quad , \quad (2a)$$

or in the condensed notation that we shall use throughout this paper, in which  $\mathbf{P} \equiv \text{Tr}P$ ,

$$\delta\mathbf{P} = \text{Tr}\frac{\delta\mathbf{P}}{\delta\mathcal{O}}\delta\mathcal{O} \quad . \quad (2b)$$

Letting  $\mathbf{L}[\{q_r\}, \{\dot{q}_r\}]$  be a trace Lagrangian that is a function of the bosonic or fermionic operators  $\{q_r\}$  and their time derivatives, and requiring that the trace action  $\mathbf{S} = \int dt\mathbf{L}$  be stationary with respect to variations of the  $q_r$ 's that preserve their bosonic or fermionic type, one finds [1] the operator Euler-Lagrange equations

$$\frac{\delta\mathbf{L}}{\delta q_r} - \frac{d}{dt}\frac{\delta\mathbf{L}}{\delta\dot{q}_r} = 0 \quad . \quad (2c)$$

Defining the momentum operator  $p_r$  conjugate to  $q_r$ , which is of the same bosonic or fermionic type as  $q_r$ , by

$$p_r \equiv \frac{\delta\mathbf{L}}{\delta\dot{q}_r} \quad , \quad (3a)$$

the trace Hamiltonian  $\mathbf{H}$  is defined by

$$\mathbf{H} = \text{Tr} \sum_r p_r \dot{q}_r - \mathbf{L} \quad . \quad (3b)$$

Performing general same-type operator variations, and using the Euler-Lagrange

equations, we find from Eq. (3b) that the trace Hamiltonian  $\mathbf{H}$  is a trace functional of the operators  $\{q_r\}$  and  $\{p_r\}$ ,

$$\mathbf{H} = \mathbf{H}[\{q_r\}, \{p_r\}] \quad , \quad (4a)$$

with the operator derivatives

$$\frac{\delta \mathbf{H}}{\delta q_r} = -\dot{p}_r \quad , \quad \frac{\delta \mathbf{H}}{\delta p_r} = \epsilon_r \dot{q}_r \quad , \quad (4b)$$

with  $\epsilon_r = 1(-1)$  according to whether  $q_r, p_r$  are bosonic (fermionic). Letting  $\mathbf{A}$  and  $\mathbf{B}$  be two trace functions of the operators  $\{q_r\}$  and  $\{p_r\}$ , it is convenient to define the *generalized Poisson bracket*

$$\{\mathbf{A}, \mathbf{B}\} = \text{Tr} \sum_r \epsilon_r \left( \frac{\delta \mathbf{A}}{\delta q_r} \frac{\delta \mathbf{B}}{\delta p_r} - \frac{\delta \mathbf{B}}{\delta q_r} \frac{\delta \mathbf{A}}{\delta p_r} \right) \quad . \quad (5a)$$

Then using the Hamiltonian form of the equations of motion, one readily finds that for a general trace functional  $\mathbf{A}[\{q_r\}, \{p_r\}]$ , the time derivative is given by

$$\frac{d}{dt} \mathbf{A} = \{\mathbf{A}, \mathbf{H}\} \quad ; \quad (5b)$$

in particular, letting  $\mathbf{A}$  be the trace Hamiltonian  $\mathbf{H}$ , and using the fact that the generalized Poisson bracket is antisymmetric in its arguments, it follows that the time derivative of  $\mathbf{H}$  vanishes.

An important property of the generalized Poisson bracket is that it satisfies [2] the Jacobi identity,

$$\{\mathbf{A}, \{\mathbf{B}, \mathbf{C}\}\} + \{\mathbf{C}, \{\mathbf{A}, \mathbf{B}\}\} + \{\mathbf{B}, \{\mathbf{C}, \mathbf{A}\}\} = 0 \quad . \quad (6a)$$

As a consequence, if  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  are two conserved charges, that is if

$$0 = \frac{d}{dt} \mathbf{Q}_1 = \{\mathbf{Q}_1, \mathbf{H}\} \quad , \quad 0 = \frac{d}{dt} \mathbf{Q}_2 = \{\mathbf{Q}_2, \mathbf{H}\} \quad , \quad (6b)$$

then their generalized Poisson bracket  $\{\mathbf{Q}_1, \mathbf{Q}_2\}$  also has a vanishing generalized Poisson bracket with  $\mathbf{H}$ , and is conserved. This is how we will use the trace dynamics formalism to get representations of the Poincaré supersymmetry algebra in the following sections.

A significant feature of trace dynamics is that, as discovered by Millard [3], the operator [3, 4]

$$\tilde{C} \equiv \sum_{r \text{ bosons}} [q_r, p_r] - \sum_{r \text{ fermions}} \{q_r, p_r\} \quad (7)$$

is conserved by the dynamics. Making the assumption (which may presuppose taking the  $N \rightarrow \infty$  limit) that trace dynamics is ergodic, one can then analyze [4] the statistical mechanics of trace dynamics for the generic case in which the conserved quantities are the trace Hamiltonian  $\mathbf{H}$  and the operator  $\tilde{C}$ . In the analysis as given in [4], the realization of fermions using the  $(-1)^F$  construction explicitly entered the argument in two places. The first was in the demonstration that trace dynamics has a generalized Liouville theorem, which is the foundation for a statistical treatment. It is easy to see that this demonstration remains valid when the fermions are realized by Grassmann odd matrices without use of the  $(-1)^F$  construction.\* The second place where the  $(-1)^F$  construction played a role was in the issue of convergence of the partition function; because  $\text{Tr}(-1)^F H$  is indefinite even

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\* In the argument at the top of p. 227 of [4], a fermionic  $\epsilon_r = -1$  was absorbed through the relation  $\epsilon_r = \epsilon_m \epsilon_n$ , with  $\epsilon_{m,n}$  the state grading factors introduced by  $(-1)^F$ . When the fermions are realized through Grassmann matrices without the  $(-1)^F$ , the state grading factors are unity and are absent, but the interchange of the order of derivatives with respect to Grassmann odd matrix elements introduces an extra minus sign in the fermionic case, again absorbing the fermionic  $\epsilon_r = -1$  and showing that the deviation of the Jacobian from unity vanishes.

when the operator Hamiltonian  $H$  has a positive definite bosonic part, it was necessary in [4] to restrict the analysis to theories in which  $\text{Tr} H$  and  $\text{Tr}(-1)^F H$  both generated the same Hamilton equations of motion, and this led to a doubling of the complexity of the statistical analysis. With Grassmann fermions this problem is avoided, since for the typical models we are studying the bosonic part of  $H$  is a positive operator, from which  $\text{Tr } \mathbf{H}$  inherits good positivity properties, and so the partition function can be expected to converge. The canonical ensemble then takes the simple form given in Eq. (48c) of [4],

$$\begin{aligned}\rho &= Z^{-1} \exp(-\text{Tr} \tilde{\lambda} \tilde{C} - \tau \mathbf{H}) \\ Z &= \int d\mu \exp(-\text{Tr} \tilde{\lambda} \tilde{C} - \tau \mathbf{H}) \quad ,\end{aligned}\tag{8}$$

with  $d\mu$  the invariant matrix (or operator) phase space measure provided by Liouville's theorem, rather than the more complicated form given in Eq. (F.1) of [4]. (As shown in [5], this canonical ensemble can also be derived from the corresponding microcanonical ensemble.) The structure of the Ward or equipartition theorems of [4] is correspondingly simplified, and leads as before to the conclusion that *the statistical mechanics of trace dynamics is complex quantum field theory*, with the average of the operator  $\tilde{C}$  playing the role of  $i\hbar$ . As suggested in [4], this means that trace dynamics behaves as a pre-quantum mechanics, in which it is likely that the ultraviolet divergences of quantum field theory are absent. Corrections to the quantum field theory approximation are expected to be of order  $\omega\tau$ , with  $\omega$  a characteristic frequency of the physics in question, and so we expect the inverse of the parameter  $\tau$  appearing in the canonical ensemble of Eq. (8), which has the dimension of mass, to play a role analogous to that of the string tension in string theories.

## 2. The Wess-Zumino Model

We begin our discussion of component field supersymmetric models with the Wess-

Zumino model. We follow the notational conventions of West [6], except that we normalize the fermion terms in the action differently, and we always use the Majorana representation for the Dirac gamma matrices. Our explicit choice of  $\gamma$  matrices is given in the Appendix, where we discuss the properties of *representation covariant*  $\gamma$  matrix identities that take a particularly simple form when expressed in Majorana representation; these will play a significant role in our analysis.

We start from the trace Lagrangian

$$\begin{aligned} \mathbf{L} = \int d^3x \text{Tr} \big( & -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \bar{\chi} \gamma^\mu \partial_\mu \chi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \\ & - m(AF + BG - \bar{\chi}\chi) \\ & - \lambda[(A^2 - B^2)F + G\{A, B\} - 2\bar{\chi}(A - i\gamma_5 B)\chi] \big) \quad , \end{aligned} \tag{9a}$$

with  $A, B, F, G$  self-adjoint bosonic  $N \times N$  matrices (or operators) and with  $\chi$  a fermionic 4 component column vector spinor, each spin component of which is a self-adjoint fermionic  $N \times N$  matrix (or operator). The notation  $\bar{\chi}$  is defined by  $\bar{\chi} = \chi^T \hat{\gamma}^0$ , with the transpose  $T$  acting only on the Dirac spinor structure, so that  $\chi^T$  is the 4 component row vector spinor constructed from the same  $N \times N$  matrices that appear in  $\chi$ , and  $\hat{\gamma}^0$  is an abbreviation for  $i\gamma^0$ . The numerical parameters  $\lambda$  and  $m$  are respectively the coupling constant and mass. Equation (9a) is identical in appearance to the usual Wess-Zumino model Lagrangian, except that we have explicitly symmetrized the term  $G\{A, B\}$ ; symmetrization of the other terms is automatic (up to total derivatives that do not contribute to the action) by virtue of the cyclic property of the trace.

Taking operator variations of Eq. (9a) by using the recipe of Eqs. (2a, b), the Euler-

Lagrange equations of Eq. (2c) take the form

$$\begin{aligned}
\partial^2 A &= mF + \lambda(\{A, F\} + \{B, G\} - 2\bar{\chi}\chi) \\
\partial^2 B &= mG + \lambda(-\{B, F\} + \{A, G\} + 2i\bar{\chi}\gamma_5\chi) \\
\gamma^\mu \partial_\mu \chi &= m\chi + \lambda(\{A, \chi\} - i\{B, \gamma_5\chi\}) \\
F &= mA + \lambda(A^2 - B^2) \\
G &= mB + \lambda\{A, B\} \quad .
\end{aligned} \tag{9b}$$

Transforming to Hamiltonian form, the canonical momenta of Eq. (3a) are

$$\begin{aligned}
p_\chi &= -\bar{\chi}\gamma^0 = i\chi^T \\
p_A &= \partial_0 A \\
p_B &= \partial_0 B \quad ,
\end{aligned} \tag{10a}$$

and the trace Hamiltonian is given by

$$\begin{aligned}
\mathbf{H} &= \int d^3x \text{Tr} \left( \frac{1}{2} [p_A^2 + p_B^2 + (\vec{\nabla} A)^2 + (\vec{\nabla} B)^2] - ip_\chi \hat{\gamma}^0 \vec{\gamma} \cdot \vec{\nabla} \chi \right. \\
&\quad \left. + \frac{1}{2} (F^2 + G^2) - m\bar{\chi}\chi + i\lambda p_\chi \hat{\gamma}^0 \{(A - i\gamma_5 B), \chi\} \right) \quad ,
\end{aligned} \tag{10b}$$

in which  $F$  and  $G$  are understood to be the functions of  $A$  and  $B$  given by the final two lines of Eq. (9b), and where we have taken care to write  $\mathbf{H}$  so that it is manifestly symmetric in the identical quantities  $p_\chi$  and  $i\chi^T$ . The trace three-momentum  $\vec{\mathbf{P}}$  is given by

$$\vec{\mathbf{P}} = - \int d^3x \text{Tr} (p_A \vec{\nabla} A + p_B \vec{\nabla} B + p_\chi \vec{\nabla} \chi) \quad , \tag{10c}$$

while the conserved operator  $\tilde{C}$  of Eq. (7) is given by

$$\tilde{C} = \int d^3x ([A, p_A] + [B, p_B] - \{\chi, p_\chi\}) \quad , \tag{10d}$$

with a contraction of the spinor indices in the final term of Eq. (10d) understood.

Let us now perform a supersymmetry variation of the fields given by

$$\begin{aligned}
\delta A &= \bar{\epsilon} \chi & \delta B &= i \bar{\epsilon} \gamma_5 \chi \\
\delta \chi &= \frac{1}{2} [F + i \gamma_5 G + \gamma^\mu \partial_\mu (A + i \gamma_5 B)] \epsilon \\
\delta F &= \bar{\epsilon} \gamma^\mu \partial_\mu \chi & \delta G &= i \bar{\epsilon} \gamma_5 \gamma^\mu \partial_\mu \chi \quad ,
\end{aligned} \tag{11}$$

with  $\epsilon$  a  $c$ -number Grassmann spinor (i.e., a four component spinor, the spin components of which are  $1 \times 1$  Grassmann matrices). Substituting Eq. (11) into the trace Lagrangian of Eq. (9a), a lengthy calculation shows that when  $\epsilon$  is constant, the variation of  $\mathbf{L}$  vanishes. The calculation parallels that done in the conventional  $c$ -number Lagrangian case, except that the trilinear cyclic identities of Eq. (1c) are used extensively in place of commutativity/anticommutativity of the fields, and the vanishing of the terms cubic in  $\chi$  is most easily established by using the cyclic property of the trace, which implies that

$$\text{Tr} \epsilon_c \chi_a \chi_b \chi_c = \text{Tr} \epsilon_c \chi_d \chi_a \chi_b = \text{Tr} \epsilon_c \chi_b \chi_d \chi_a \quad , \tag{12a}$$

together with the cyclic identity valid for Majorana representation  $\gamma$  matrices (see the Appendix),

$$\sum_{\text{cycle } a \rightarrow b \rightarrow d \rightarrow a} [\hat{\gamma}_{ab}^0 \hat{\gamma}_{cd}^0 + (\hat{\gamma}^0 \gamma_5)_{ab} (\hat{\gamma}^0 \gamma_5)_{cd}] = 0 \quad . \tag{12b}$$

When  $\epsilon$  is not constant, the variation of  $\mathbf{L}$  is given by

$$\begin{aligned}
\delta \mathbf{L} &= \int d^3 x \text{Tr} (\bar{J}^\mu \partial_\mu \epsilon) \\
\bar{J}^\mu &= - \bar{\chi} \gamma^\mu [(\gamma^\nu \partial_\nu + m)(A + i \gamma_5 B) + \lambda(A^2 - B^2 + i \gamma_5 \{A, B\})] \quad ,
\end{aligned} \tag{13a}$$

which identifies the trace supercharge  $\mathbf{Q}_\alpha$  as

$$\begin{aligned}
\mathbf{Q}_\alpha &\equiv \int d^3 x \text{Tr} \bar{J}^0 \alpha \\
&= \int d^3 x \text{Tr} \frac{1}{2} (p_\chi + i \chi^T) [(\gamma^\nu \partial_\nu + m)(A + i \gamma_5 B) + \lambda(A^2 - B^2 + i \gamma_5 \{A, B\})] \alpha \quad ,
\end{aligned} \tag{13b}$$

where we have again taken care to express  $\mathbf{Q}_\alpha$  symmetrically in the identical quantities  $p_\chi$  and  $i\chi^T$ . It is straightforward to check, using the equations of motion and the cyclic identity, that  $\text{Tr}\bar{J}^\mu$  is a conserved trace supercurrent, which implies that the trace supercharge is conserved.

We are now ready to check the closure of the supersymmetry algebra under the generalized Poisson bracket of Eq. (5a), which for the Hamiltonian dynamics of the Wess-Zumino model gives

$$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\} = \text{Tr}\left[\frac{\delta\mathbf{Q}_\alpha}{\delta A}\frac{\delta\mathbf{Q}_\beta}{\delta p_A} + \frac{\delta\mathbf{Q}_\alpha}{\delta B}\frac{\delta\mathbf{Q}_\beta}{\delta p_B} - \frac{\delta\mathbf{Q}_\alpha}{\delta\chi}\frac{\delta\mathbf{Q}_\beta}{\delta p_\chi} - (\alpha \leftrightarrow \beta)\right] . \quad (14a)$$

There are two strategies for carrying out the considerable amount of algebra involved in evaluating Eq. (14a). The first is to directly rearrange into the expected form, verifying along the way various Majorana representation  $\gamma$  matrix identities that are needed; the second is to first Fierz transform so as to isolate a factor of the form  $\alpha^T\Gamma\beta$ , and then to show that this yields the expected result. We shall use the first method here, and the second method in discussing the supersymmetric Yang-Mills model in the next section. Proceeding by the first method, we find that Eq. (14) rearranges, using the cyclic identities of Eq. (1c), into the form

$$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\} = \bar{\alpha}\gamma^0\beta\mathbf{H} - \bar{\alpha}\vec{\gamma}\beta \cdot \vec{\mathbf{P}} \quad , \quad (14b)$$

with  $\mathbf{H}$  and  $\vec{\mathbf{P}}$  the trace Hamiltonian and three-momentum given above. The  $\gamma$  matrix identities needed can be obtained by repeated applications either of the cyclic identity of Eq. (12b), or of the additional identity (with  $\ell, m, n$  spatial indices, and  $\epsilon_{\ell mn}$  the three index antisymmetric tensor with  $\epsilon_{123} = 1$ )

$$\begin{aligned} & \gamma_{ab}^\ell \hat{\gamma}_{cd}^0 + \gamma_{db}^\ell \hat{\gamma}_{ca}^0 - (\gamma^\ell \gamma_5)_{ab} (\hat{\gamma}^0 \gamma_5)_{cd} - (\gamma^\ell \gamma_5)_{db} (\hat{\gamma}^0 \gamma_5)_{ca} \\ &= \delta_{ad} (\hat{\gamma}^0 \gamma_\ell)_{bc} - (\hat{\gamma}^0 \gamma_\ell)_{ad} \delta_{bc} + \epsilon_{\ell mn} (\gamma_\ell \gamma_m \gamma_5)_{ad} (\gamma_\ell \gamma_n)_{cb} \quad , \end{aligned} \quad (15)$$

which we have verified by the method described in the Appendix. It is also easy to check that  $\mathbf{Q}_\epsilon$  plays the role of the generator of supersymmetry transformations for the dynamical variables  $A, B, \chi$  under the generalized Poisson bracket, since we readily find (for constant Grassmann even parameters  $a, b$  and Grassmann odd parameter  $c$ )

$$\{\text{Tr}(aA + bB + c\chi), \mathbf{Q}_\epsilon\} = \text{Tr}(a\delta A + b\delta B + c\delta\chi) \quad , \quad (16)$$

with  $\delta A, \delta B, \delta\chi$  the supersymmetry variations given by Eq. (11) above, after elimination of the auxiliary fields  $F, G$  by their equations of motion.

### 3. The Supersymmetric Yang-Mills Model

As our next example of a component field supersymmetric model, we discuss supersymmetric Yang-Mills theory. (In Ref. [7] we have given a simpler analog of this discussion, in the context of the matrix model for M theory.) We start from the trace Lagrangian

$$\mathbf{L} = \int d^3x \text{Tr} \left[ \frac{1}{4g^2} F_{\mu\nu}^2 - \bar{\chi} \gamma^\mu D_\mu \chi + \frac{1}{2} D^2 \right] \quad , \quad (17a)$$

with the field strength  $F_{\mu\nu}$  and covariant derivative  $D_\mu$  constructed from the gauge potential  $A_\mu$  according to

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\ D_\mu \mathcal{O} &= \partial_\mu \mathcal{O} + [A_\mu, \mathcal{O}] \end{aligned} \quad (17b)$$

$$\Rightarrow D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0 \quad .$$

In Eq. (17b), the potential components  $A_\mu$  are each an anti-self-adjoint, and the auxiliary field  $D$  a self-adjoint, bosonic  $N \times N$  matrix (or operator), and each spinor component of  $\chi$  is a self-adjoint fermionic  $N \times N$  matrix (or operator). The Euler-Lagrange equations of

motion are

$$D = 0$$

$$\gamma^\mu D_\mu \chi = 0 \quad (18a)$$

$$D_\mu F^{\mu\nu} = 2g^2 \bar{\chi} \gamma^\nu \chi \quad ;$$

as usual for a gauge system, the  $\nu = 0$  component of Eq. (18a) is not a dynamical evolution equation, but rather the constraint

$$D_\ell F^{\ell 0} = 2g^2 \bar{\chi} \gamma^0 \chi \quad . \quad (18b)$$

Going over to the Hamiltonian formalism, the canonical momenta are given by

$$p_{A_\ell} = -\frac{1}{g^2} F_{0\ell} \quad , \quad p_\chi = i\chi^T \quad , \quad (19a)$$

and the axial gauge trace Hamiltonian (see [1] for a derivation and references) is

$$\mathbf{H} = \mathbf{H}_A + \mathbf{H}_\chi \quad , \quad (19b)$$

with

$$\begin{aligned} \mathbf{H}_A &= \int d^3x \text{Tr} \left( \frac{-g^2}{2} \sum_{\ell=1}^2 p_{A_\ell}^2 - \frac{1}{2g^2} F_{03}^2 \right. \\ &\quad \left. - \frac{1}{2g^2} (\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2])^2 - \frac{1}{2g^2} [(\partial_3 A_1)^2 + (\partial_3 A_2)^2] \right) \\ F_{03} &= \frac{1}{2} g^2 \int_{-\infty}^{\infty} dz' \epsilon(z - z') [-(p_\chi \chi + \chi^T p_\chi^T) + D_1 p_{A_1} + D_2 p_{A_2}]|_{z'} \\ \mathbf{H}_\chi &= -i \int d^3x \text{Tr} (p_\chi \hat{\gamma}^0 \gamma_\ell D_\ell \chi) \quad , \end{aligned} \quad (19c)$$

where we have taken care to write  $\mathbf{H}$  in a form symmetric in the identical quantities  $p_\chi$  and  $i\chi^T$ , and where  $\epsilon(z) = 1(-1)$  for  $z > 0(z < 0)$ . The trace three momentum is

$$\mathbf{P}_m = - \int d^3x \text{Tr} \left( \sum_{\ell=1}^3 F_{m\ell} p_{A_\ell} + p_\chi D_m \chi \right) \quad , \quad (20a)$$

and the conserved operator  $\tilde{C}$  of Eq. (7) is given by

$$\tilde{C} = \int d^3x \left( \sum_{\ell=1}^2 [A_\ell, p_{A_\ell}] - \{\chi, p_\chi\} \right) \quad , \quad (20b)$$

with a contraction of the spinor indices in the final term of Eq. (20b) understood. By virtue of the constraint of Eq. (18b), the conserved operator  $\tilde{C}$  can also be written as

$$\tilde{C} = - \int d^3x \sum_{\ell=1}^2 \partial_\ell p_{A_\ell} = - \int_{\text{sphere at } \infty} d^2S_\ell p_{A_\ell} \quad , \quad (20c)$$

which vanishes when the surface integral in Eq. (20c) is zero.

Making now the supersymmetry variations

$$\begin{aligned} \delta A_\mu &= i g \bar{\epsilon} \gamma_\mu \chi \\ \delta \chi &= \left( \frac{i}{8g} [\gamma_\mu, \gamma_\nu] F^{\mu\nu} + \frac{i}{2} \gamma_5 D \right) \epsilon \\ \delta D &= i \bar{\epsilon} \gamma_5 \gamma^\mu D_\mu \chi \quad , \end{aligned} \quad (21a)$$

in the trace Lagrangian, we find that when  $\epsilon$  is constant, the variation vanishes. Again the calculation parallels that done in the  $c$ -number Lagrangian case, except that the trilinear cyclic identities of Eq. (1c) are used in place of commutativity/anticommutativity of the fields, and the vanishing of terms cubic in  $\chi$  is most easily established by using Eq. (12a) and the cyclic identity valid for Majorana representation  $\gamma$  matrices (see the Appendix)

$$\sum_{\text{cycle } a \rightarrow b \rightarrow d \rightarrow a} (\hat{\gamma}^0 \gamma^\mu)_{ab} (\hat{\gamma}^0 \gamma_\mu)_{cd} = 0 \quad . \quad (21b)$$

When  $\epsilon$  is not a constant, the variation of  $\mathbf{L}$  is given by

$$\begin{aligned} \delta \mathbf{L} &= \int d^3x \text{Tr}(\bar{J}^\mu \partial_\mu \epsilon) \\ \bar{J}^\mu &= - \frac{i}{4g} \bar{\chi} \gamma^\mu F_{\nu\sigma} [\gamma^\nu, \gamma^\sigma] \quad , \end{aligned} \quad (22a)$$

from which we construct the trace supercharge  $\mathbf{Q}_\alpha$  as

$$\mathbf{Q}_\alpha = \int d^3x \text{Tr} \frac{i}{8g} (p_\chi + i \chi^T) F_{\nu\sigma} [\gamma^\nu, \gamma^\sigma] \alpha \quad . \quad (22b)$$

Again, it is straightforward to check, using the equations of motion and the cyclic identity, that  $\text{Tr} \bar{J}_\mu$  is a conserved trace supercurrent, which implies that the trace supercharge is conserved.

We are now ready to check the closure of the supersymmetry algebra under the generalized Poisson bracket of Eq. (5a), which for the Hamiltonian dynamics of the supersymmetric Yang-Mills model gives

$$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\} = \text{Tr} \left[ \sum_{\ell=1}^2 \frac{\delta \mathbf{Q}_\alpha}{\delta A_\ell} \frac{\delta \mathbf{Q}_\beta}{\delta p_{A_\ell}} - \sum_{d=1}^4 \frac{\delta \mathbf{Q}_\alpha}{\delta \chi^d} \frac{\delta \mathbf{Q}_\beta}{\delta p_{\chi^d}} - (\alpha \leftrightarrow \beta) \right] . \quad (23a)$$

We proceed now by the Fierz transformation method mentioned in Sec. 2 above. We begin by rewriting the boson terms in Eq. (23a) as

$$\int d^3x \text{Tr} \sum_{\ell=1}^2 f_\ell^a \alpha_a \bar{\beta}^c g_{\ell c} - (\alpha \leftrightarrow \beta) \quad , \quad (23b)$$

and the fermion terms as

$$\int d^3x \text{Tr} \sum_{d=1}^4 h^{da} \alpha_a \bar{\beta}^c k_{dc} - (\alpha \leftrightarrow \beta) \quad , \quad (23c)$$

with the coefficient functions  $f, g, h, k$  readily determined once the operator variations in Eq. (23a) have been computed. Performing a Fierz transformation by using Eq. (A.80) of West [6] then shows that verifying the supersymmetry algebra of Eq. (14b), with  $\mathbf{Q}_{\alpha, \beta}$ ,  $\mathbf{H}$  and  $\vec{\mathbf{P}}$  now given by the Yang-Mills expressions of this section, is equivalent to verifying the three identities

$$\begin{aligned} (... \gamma^0 ...) &= \frac{1}{2} \mathbf{H} \\ (... \gamma^m ...) &= \frac{1}{2} \mathbf{P}^m \\ (... [\gamma^\mu, \gamma^\nu] ...) &= 0 \quad , \end{aligned} \quad (24a)$$

with

$$(... \Gamma ...) \equiv - \int d^3x \frac{1}{4} \text{Tr} \left( \sum_{\ell=1}^2 f_\ell^a \Gamma_a^d g_{\ell d} + \sum_{d=1}^4 h^{da} \Gamma_a^c k_{dc} \right) . \quad (24b)$$

Equation (24b) contains both local terms, and nonlocal terms that couple variables at differing values of  $z$ . The nonlocal terms are found to vanish identically when the constraint equation and symmetries of the integrands are taken into account, while the local terms are seen, by an enumeration of cases, to obey Eq. (24a). Examining the role of the supercharge as a generator of transformations, in analogy with Eq. (16), in the Yang-Mills case the supercharge is found to generate the supersymmetry variations of Eq. (21a), plus an infinitesimal change of gauge.

We conclude by showing how the results of this section include the conventional case of  $U(M)$  supersymmetric Yang-Mills theories, and why at the same time they are more general. Consider the case in which the  $N \times N$  matrices acted on by  $\text{Tr}$  have dimension  $N = MP$ , and expand the matrices on a complete basis of  $U(M)$  matrices  $\lambda_i$ , so that for the potential  $A_\mu$  we have

$$A_\mu = \sum_{i=0}^{M^2} \frac{1}{2} \lambda_i A_\mu^i \quad , \quad (25a)$$

where the coefficients  $A_\mu^i$  are now  $P \times P$  matrices. Then the commutator term  $[A_\mu, A_\nu]$  in Eq. (17b) becomes

$$[A_\mu, A_\nu] = \frac{1}{2} \sum_{ij} ([\lambda_i, \lambda_j] \{A_\mu^i, A_\nu^j\} + \{\lambda_i, \lambda_j\} [A_\mu^i, A_\nu^j]) \quad . \quad (25b)$$

When  $P = 1$ , so that the  $A_\mu^i$  all commute, the second term on the right hand side of Eq. (25b) vanishes, and it reduces to the conventional expression for the commutator term in a Yang-Mills theory. However, our formalism generalizes this conventional model to allow any  $P > 1$ , including the limit  $P \rightarrow \infty$ , in which case the second term in Eq. (25b) contributes as well as the first. Similar remarks apply to the other matrix commutators appearing in the derivations of this section.

#### 4. Superspace Considerations and Discussion

The derivations of Secs. 2 and 3 have all been carried out in the component formalism, which requires doing a separate computation for each Poincaré supersymmetry multiplet. However, there is a simple and general superspace argument for the results we have obtained. Recall that superspace is constructed by introducing four fermionic coordinates  $\theta_\alpha$  corresponding to the four space-time coordinates  $x_\mu$ . The graded Poincaré algebra is then represented by differential operators constructed from the superspace coordinates, and superfields are represented by finite polynomials in the fermionic coordinates  $\theta_\alpha$ , with coefficient functions that depend on  $x_\mu$ . To generalize the superspace formulation to give trace dynamics models, one simply replaces these coefficient functions by  $N \times N$  matrices (or operators), and one inserts a trace  $\text{Tr}$  acting on the superspace integrals used to form the action. Then the standard argument that the action is invariant under superspace translations still holds for the trace action formed this way from the matrix components of the superfields. We immediately see from this argument why it is essential for the supersymmetry parameter  $\epsilon$  to be a  $c$ -number and not also a matrix; this parameter appears as the magnitude of an infinitesimal superspace translation, and since the superspace coordinates  $x_\mu$  and  $\theta_\alpha$  are  $c$ -numbers, the parameter  $\epsilon$  must be one also.

The construction just given gives reducible supersymmetry representations, and various constraints must be applied to the superfields to pick out irreducible representations. Since these constraints act linearly on the expansion coefficients, they can all be immediately generalized (with the usual replacement of complex conjugation for  $c$ -numbers by the adjoint) to the case in which the coefficient functions are matrices or operators.

The simplicity of this argument suggests that for all nonextended rigid supersym-

metry theories for which there exists a superspace construction, there should exist trace dynamics generalizations, with component field forms analogous to those presented above and with corresponding representation covariant  $\gamma$  matrix identities. In this paper we have not dealt with either extended supersymmetries, or with locally supersymmetric theories; these will be the subject of further investigations into supersymmetric trace dynamics theories.

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### Appendix: Gamma Matrix Conventions and Identities

We work with Majorana representation  $\gamma$  matrices constructed explicitly as follows. Let  $\sigma_{1,2,3}$  and  $\tau_{1,2,3}$  be two independent sets of Pauli spin matrices; then we take

$$\begin{aligned}
\gamma^0 &= -\gamma_0 = -i\sigma_2\tau_1 \\
\hat{\gamma}^0 &= i\gamma^0 = \sigma_2\tau_1 \\
\gamma^1 &= \gamma_1 = \sigma_3 \\
\gamma^2 &= \gamma_2 = -\sigma_2\tau_2 \\
\gamma^3 &= \gamma_3 = -\sigma_1 \\
\gamma_5 &= i\gamma^1\gamma^2\gamma^3\gamma^0 = -\sigma_2\tau_3 \\
\hat{\gamma}^0\gamma_5 &= i\tau_2 \quad ,
\end{aligned}
\tag{A.1a}$$

so that  $\hat{\gamma}^0, \gamma_5, \hat{\gamma}^0 \gamma_5$  are skew symmetric and  $\gamma^1, \gamma^2, \gamma^3$  are symmetric, and

$$\hat{\gamma}^0 \gamma^{\mu T} \hat{\gamma}^0 = -\gamma^\mu \quad . \quad (A.1b)$$

For this choice of  $\gamma$  matrices, the four matrices  $\gamma^\mu$  are real.

The identities of Eqs. (12b), (15), and (21b) are easily verified by the following method. Replace each four component spinor index  $a, b, c, d$  by a pair of two component spinor indices  $AA', BB', CC', DD'$ , with the unprimed indices on the matrices  $\sigma$  and the primed indices on the matrices  $\tau$ . Then each identity takes the form

$$\Delta_{AA'BB'CC'DD'} = 0 \quad , \quad (A.2a)$$

with Eq. (A.2) symmetric under the simultaneous interchange

$$A \leftrightarrow B, \quad A' \leftrightarrow B' \quad , \quad (A.2b)$$

for the identities of Eq. (15) and (21b) (which are symmetric under  $a \leftrightarrow b$ ) and antisymmetric under this interchange for the identity of Eq. (12b) (which is antisymmetric under  $a \leftrightarrow b$ ). To verify an identity, it suffices to verify the vanishing of its contraction with a complete set of 16 projectors on the  $4 \times 4$  matrix with indices  $a, b$ , which in terms of  $\sigma$  and  $\tau$  are

$$[\delta_{AB}, (\sigma_{1,2,3})_{AB}] \otimes [\delta_{A'B'}, (\tau_{1,2,3})_{A'B'}] \quad , \quad (A.3)$$

10 of which are symmetric and 6 of which are antisymmetric under the interchange of Eq. (A.2b). Thus 10 contractions must be done to verify the identities of Eqs. (15) and (21b), and 6 contractions to verify the identity of Eq. (12b); these are readily done since the projectors involve two factors which repeat in different combinations, and since the contractions for individual factors involve only Pauli matrix arithmetic.

The identities of Eqs. (12b), (15), and (21b) are representation covariant, in that they do not take the same form in representations in which the Dirac gamma matrices are complex rather than real. To see this, we note that the matrices in a general representation  $\gamma_G^\mu$  are related to the Majorana representation matrices  $\gamma^\mu$  given above by

$$\gamma_G^\mu = U^\dagger \gamma^\mu U = U^{T*} \gamma^\mu U \quad , \quad (A.4)$$

with  $U$  a unitary matrix which in general is complex, as a result of which the row and column indices transform with different matrices. However, the identities used in the text mix row and column indices; for example, in Eq. (12b) there is one term in the cyclic sum in which  $a$  is a row index, and two terms in which  $a$  is a column index. (By way of contrast, the more familiar Fierz identities only interchange two row indices, and so do not mix row and column indices.) Hence we cannot get a representation invariant form of the identity by two applications of Eq. (A.4), since in the second and third terms of the cyclic sum, we will have a row index contracted with a  $U$  and a column index contracted with a  $U^*$ , which does not correspond with Eq. (A4). However, we can easily get a representation covariant form of Eq. (12b) by contracting all indices with a  $U^*$ , and wherever  $U^*$  contracts with a column index using the identity

$$U^* = UU^{*T}U^* = U\gamma^* \quad , \quad \gamma \equiv U^T U \quad , \quad (A.5)$$

with  $\gamma$  a matrix which appears on pp. 341-342 of the book cited in Ref. [1] (because it plays a role in the transformation properties of the Dirac equation in quaternionic quantum mechanics). We can then apply Eq. (A.4) to all the gamma matrices, giving for Eq. (12b),

for example, the representation covariant form

$$\sum_{\text{cycle } a \rightarrow b \rightarrow d \rightarrow a} [(\hat{\gamma}^0 \gamma^*)_{ab} (\hat{\gamma}^0 \gamma^*)_{cd} + (\hat{\gamma}^0 \gamma_5 \gamma^*)_{ab} (\hat{\gamma}^0 \gamma_5 \gamma^*)_{cd}] = 0 \quad . \quad (A.6)$$

For a change of representation which preserves reality of the  $\gamma$  matrices, we have  $U^* = U$ ,  $\gamma = U^T U = U^{*T} U = 1$ , and Eq. (A.6) is identical to Eq. (12b), but for general changes of representation

the identity is form covariant but not form invariant.

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